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LETTER TO THE EDITOR

New inverses of the attenuated Abel integral equation

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Abstract. Abel's integral equation relates the line-of-sight radiance to the emission coefficient distribution function of an extended, cylindrically symmetric transparent radiation source. In the presence of absorption this equation is modified. We present here two new inversion formulae for this attenuated Abel equation for the case of constant absorption throughout the source. As one of them is completely derivative-free and the other requires differentiation of a weighted integral over the radiance rather than a differentiation of the radiance function itself, they are particularly well suited for use with *measured* radiance data, where differentiation invariably results in a very large amplification of the random experimental error inherent in the data. An illustrative example is also given.

Inversion of Abel's integral equation (Anderssen 1973, Buck 1974, Cremers and Birkeback 1966) is an essential step in the analysis of measured data in numerous fields of research. In flame and plasma diagnostics (Barr 1962), for example, Abel's equation relates the measured line-of-sight radiance of a cylindrically symmetric source to its emission coefficient distribution. If the source is optically thin so that no absorption occurs within the source, the measured radiance data can be Abel-inverted to obtain the physically important emission coefficient distribution. In this letter an extension of the theory of Abel-type equations to the case of a non-zero constant absorption in the source is presented. Exact analytic formulae for the attenuated Abel equation are derived and discussed.

Consider a cylindrical radiation source the cross section of which is of radius unity and is shown in figure 1. Assume that both the absorption and the emission coefficient distributions, $\mu(r)$ and $g(r)$ respectively, are cylindrically symmetric inside the source and zero outside. Then the line-of-sight radiance measured by the detector is given by

$$I(y)=\int_{-\infty}^{\infty} \mathrm{d}x\, g(r)\exp\left(-\int_{-\infty}^{x} \mathrm{d}x'\, \mu(r')\right).$$

For the case at hand $\mu(r)=\mu$ is constant and since $a=x=(r^2-y^2)^{1/2}$ and $b=(1-y^2)^{1/2}$ we obtain

$$I(y)=2\int_{y}^{1} g(r)r\exp[-\mu(1-y^2)]\cosh[\mu(r^2-y^2)^{1/2}](r^2-y^2)^{-1/2}\,\mathrm{d}r. \qquad (1)$$

This is the attenuated Abel equation (Barrett 1982, 1984, Clough and Barrett 1983).

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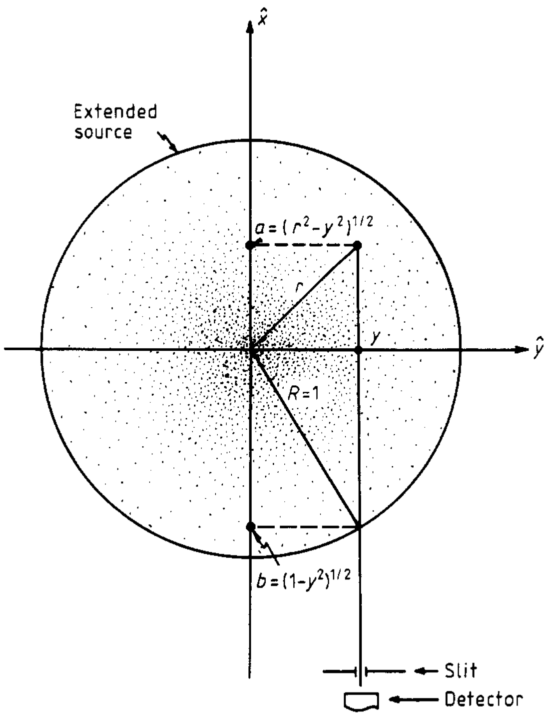

Figure 1. Cross section of experimental set-up.

For the case of zero absorption (1) reduces to the usual Abel equation

$$I(y)=2\int_y^1 g(r)r(r^2-y^2)^{-1/2}\,\mathrm{d}r \tag{2}$$

for which three analytic inverses are available (Chan and Lu 1981, Whittaker and Watson 1948, Deutsch and Beniaminy 1982). If, however, $\mu(r)$ is non-constant, no analytic inverse of the general equation is known. Recently, Clough and Barrett (1983) obtained an inverse of (1). If this equation is written as

$$F(y)=2\int_y^1 g(r)r\cosh[\mu(r^2-y^2)^{1/2}](r^2-y^2)^{-1/2}\,\mathrm{d}r \tag{3}$$

where

$$F(y)=I(y)\exp[\mu(1-y^2)^{1/2}], \tag{4}$$

then the inverse is given by

$$g^I(r)=-(1/\pi)\int_r^1 F'(y)\cos[\mu(y^2-r^2)^{1/2}](r^2-y^2)^{-1/2}\,\mathrm{d}y \tag{5}$$

where the prime denotes differentiation with respect to y.

In practice the radiance function is measured at a finite number of discrete points $I(y_i)$ $(i=1,2,\ldots,N)$. Thus, inversion of (3) by $g^I(r)$ necessarily involves differentiating experimental data, a process notorious for its enhancement of the random errors inherent in all measured data. A 200-fold amplification of such errors was found

(Deutsch and Beniaminy 1982) to occur in some cases due to differentiation when inverting the usual Abel equation, (2), with measured $I(y_i)$ data. Although rather effective numerical methods for minimising error amplification are available (Minerbo and Levy 1969, Anderssen 1976, Deutsch and Beniaminy 1983, Deutsch 1983) it is preferable to eliminate the problem altogether by using inversion formulae which do not require differentiating $I(y)$.

To derive such an inverse, we integrate (5) by parts to obtain

$$g^{II}(r) = -(1/\pi)[F(y)\cos[\mu(y^2-r^2)^{1/2}](y^2-r^2)^{-1/2}]_r^1 + (1/\pi)\left\{-\mu\int_r^1 F(y)\sin[\mu(y^2-r^2)^{1/2}]y(y^2-r^2)^{-1}\,\mathrm{d}y - \int_r^1 F(y)\cos[\mu(y^2-r^2)^{1/2}]y(y^2-r^2)^{-3/2}\,\mathrm{d}y\right\}. \tag{6}$$

Since $F(y)$ is well behaved, we may write

$$\begin{aligned}[F(y)\cos[\mu(y^2-r^2)^{1/2}](y^2-r^2)^{-1/2}]_r^1 &= F(1)\cos[\mu(1-r^2)^{1/2}](1-r^2)^{-1/2} + F(r)\cos[\mu(1-r^2)^{1/2}][(y^2-r^2)^{-1/2}]_r^1 \\ &\quad - F(r)\cos[\mu(1-r^2)^{1/2}](1-r^2)^{-1/2} \\ &= [F(1)-F(r)]\cos[\mu(1-r^2)^{1/2}](1-r^2)^{-1/2} - F(r)\cos[\mu(1-r^2)^{1/2}] \\ &\quad \times\int_r^1 y(y^2-r^2)^{-3/2}\,\mathrm{d}y.\end{aligned} \tag{7}$$

Substituting (7) into (6) and using the notation

$$\begin{aligned}h(y) &= F(y)\cos[\mu(y^2-r^2)^{1/2}] \\ f(y) &= F(y)\{\cos[\mu(y^2-r^2)^{1/2}] + \mu(y^2-r^2)^{1/2}\sin[\mu(y^2-r^2)^{1/2}]\}\end{aligned} \tag{8}$$

we finally obtain

$$g^{II}(r) = -(1/\pi)[h(1)-h(r)](1-r^2)^{-1/2} - (1/\pi)\int_r^1 [f(y)-f(r)]y(y^2-r^2)^{-3/2}\,\mathrm{d}y. \tag{9}$$

This is the required derivative-free inversion formula of the attenuated Abel equation, (3). The integrable singularity of the integrand at $y = r$ can be taken care of by standard analytic methods (Davies and Rabinowitz 1975) or, if the integral is evaluated numerically, simply by using a procedure not requiring the value of the integrand at the limit of integration, such as the adaptive three-point Gaussian method of Robinson (1971).

A third inverse of the attenuated Abel equation can be obtained by taking first the Fourier cosine transform of (3), changing the order of the two integrals and then performing an inverse Hankel transform. As shown by Clough and Barrett (1983) this yields

$$g^{III}(r) = (1/\pi)\int_0^\infty \nu J_0(2\pi r\nu)\,\mathrm{d}\nu\int_0^\infty \cos\{2\pi y[\nu^2+(\mu/2\pi)^2]^{1/2}\}F(y)\,\mathrm{d}y \tag{10}$$

where ν is the Fourier-conjugate variable of y and J_0 is the zeroth-order Bessel function.

Using the recurrence relations (Gradshteyn and Ryzhik 1980)

$$J_0'(z) = -J_1(z), \qquad J_1(z) + zJ_1'(z) = zJ_0(z)$$

where the prime denotes the derivative, we find

$$-\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}\nu}J_0(2\pi r\nu) = (2\pi r\nu)J_0(2\pi r\nu).$$

Substitution in (10) yields

$$g^{III}(r) = -\frac{1}{4\pi^3 r}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^\infty F(y)\,\mathrm{d}y\int_0^\infty\left[\frac{\mathrm{d}}{\mathrm{d}\nu}J_0(2\pi r\nu)\right]\cos\{2\pi y[\nu^2+(\mu/2\pi)^2]^{1/2}\}\,\mathrm{d}\nu.$$

Integrating the second integral by parts we obtain

$$g^{III}(r) = -\frac{1}{4\pi^3 r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\int_0^\infty F(y)\cos(\mu y)\,\mathrm{d}y\right]$$
$$-\frac{1}{2\pi^2 r}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^\infty F(y)y\,\mathrm{d}y\int_0^\infty \nu J_0(2\pi r\nu)\sin\{2\pi y[\nu^2+(\mu/2\pi)^2]^{1/2}\}$$
$$\times[\nu^2+(\mu/2\pi)^2]^{-1/2}\,\mathrm{d}\nu.$$

The first term is identically zero, while for the second (Erdelyi 1954)

$$\int_0^\infty \nu J_0(2\pi r\nu)\sin\{2\pi y[\nu^2+(\mu/2\pi)^2]^{1/2}\}[\nu^2+(\mu/2\pi)^2]^{-1/2}\,\mathrm{d}\nu$$
$$= 2\pi\cos[\mu(y^2-r^2)^{1/2}](y^2-r^2)^{-1/2} \qquad |r|<y.$$

Thus, we finally obtain

$$g^{III}(r) = -(\pi r)^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^1 F(y)y\cos[\mu(y^2-r^2)^{1/2}](y^2-r^2)^{-1/2}\,\mathrm{d}y. \qquad (11)$$

Note that while here differentiation *is* required, it does not operate directly on the measured data but rather on a weighted integral thereof. Thus, g^{III} is less prone to error amplification than g^I but more so than g^{II}. It should also be noted that since for $\mu = 0$

$$h(y) = f(y) = F(y) = I(y),$$

the three inverses of the attenuated Abel equation, g^I of Clough and Barrett (1983) and g^{II} and g^{III} derived here, reduce to the three known inverses (Whittaker and Watson 1948, Deutsch and Beniaminy 1982) of the usual Abel equation

$$g^I(r) \to -(1/\pi)\int_r^1 I'(y)(y^2-r^2)^{-1/2}\,\mathrm{d}y$$

$$g^{II}(r) \to -(1/\pi)[I(1)-I(r)](1-r^2)^{-1/2}$$
$$+\int_r^1 [I(y)-I(r)]y(y^2-r^2)^{-3/2}\,\mathrm{d}y$$

$$g^{III}(r) \to -(\pi r)^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^1 I(y)y(y^2-r^2)^{-1/2}\,\mathrm{d}y$$

as expected.

Finally, as an illustration of the above results, consider the case of a constant emission coefficient throughout the source

$$g(r)=\begin{cases}a & r<1\\ 0 & r>1.\end{cases}$$

$F(y)$ is then given by (Clough and Barrett 1983)

$$F(y)=(2a/\mu)\sinh[\mu(1-y^2)^{1/2}].$$

Substitution in (11) yields

$$g^{III}(r)=-(2a/\mu\pi r)\frac{\mathrm{d}}{\mathrm{d}r}\int_r^1 \sinh[\mu(1-y^2)^{1/2}]y$$

$$\times\cos[\mu(y^2-r^2)^{1/2}](y^2-r^2)^{-1/2}\,\mathrm{d}y$$

$$=-(2a/\mu^2\pi r)\frac{\mathrm{d}}{\mathrm{d}r}\int_r^1 \sinh[\mu(1-y^2)^{1/2}]\frac{\mathrm{d}}{\mathrm{d}r}\{\sin[\mu(y^2-r^2)^{1/2}]\}\,\mathrm{d}y.$$

Integrating by parts and denoting $\omega^2=\mu^2(y^2-r^2)$ we obtain

$$g^{III}(r)=-(2a/\mu\pi r)\frac{\mathrm{d}}{\mathrm{d}r}\int_0^{\mu(1-r^2)^{1/2}} \cosh\{[\mu^2(1-r^2)-\omega^2]^{1/2}\}$$

$$\times[\mu^2(1-r^2)-\omega^2]^{-1/2}\omega\sin\omega\,\mathrm{d}\omega.$$

Noting that

$$-\int_0^\infty \omega f(\omega)\sin\omega\,\mathrm{d}\omega=-\left(\int_0^\infty \omega f(\omega)\sin(\omega z)\,\mathrm{d}\omega\right)_{z=1}$$

$$=\left(\frac{\mathrm{d}}{\mathrm{d}z}\int_0^\infty f(\omega)\cos(\omega z)\,\mathrm{d}\omega\right)_{z=1}$$

and using a table of Fourier cosine transforms (Erdelyi 1954) we obtain

$$g^{III}(r)=(2a/\mu^3\pi r)(\mathrm{d}/\mathrm{d}r)[(\mathrm{d}/\mathrm{d}z)(\pi/2)J_0[\mu(1-r^2)^{1/2}(z^2-1)^{1/2}]]_{z=1}.$$

Differentiating with respect to z, replacing J_0' first by $-J_1$ and then by its expansion in powers of (z^2-1) we obtain

$$g^{III}(r)=-(a/r)\,(\mathrm{d}/\mathrm{d}r)[\tfrac{1}{2}(1-r^2)-(\mu^2/16)(1-r^2)^2(z^2-1)+\ldots]_{z=1}$$

which, for $z=1$, yields $g^{III}(r)=a$ as required. The same result is obtained using g^{II} of (9).

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